Equilibrium measure, Poisson kernel and effective resistance on networks

Enrique Bendito, Ángeles Carmona and Andrés M. Encinas

Abstract. We consider the Laplacian of a finite network as a kernel on the vertex set. The properties of this kernel allow us to assign to every proper set an equilibrium measure and a capacity. So, we can build a discrete Potential Theory with respect to the Laplacian kernel on networks. We aim here at showing how equilibrium measures can be used to obtain simple expressions for both Poisson and Green kernels and hence to deduce nice expressions for the effective resistance and the hitting time.

1 Introduction

A systematic treatment of Potential Theory with respect to a kernel can be found in the work of B. Fuglede [7]. Although in that work the particular case in which the underlying space is finite was not considered explicitly, the main results in this context can be deduced without difficulty from the general case. The finite case was explicitly considered in classical works of G. Choquet and J. Deny, for instance, see [4]. However, the kernels considered in all these papers are always non negative, which corresponds to developing Potential Theory with respect to the Green kernel.

In the present work we deal with the Potential Theory of a signed kernel on a finite network. Since any linear operator on a finite space can be considered as an integral operator, the Laplace operator of a network can be interpreted as a kernel on the vertex set. This approach (see [2, 3]) differs from both the classical Potential Theory with respect to the Green kernel and from the Dirichlet Forms Theory.

In Section 2 we present, for the sake of completeness, some of the results on the Discrete Potential Theory with respect to the Laplacian kernel obtained in [2, 3]. Essentially, these results follow from the fact that this kernel satisfies two fundamental principles, namely the energy and Frostman's maximum principles, which allow us to solve equilibrium problems whose solutions will be the basic tool in the rest of the work. Moreover, the Wiener capacity with respect to the Laplacian kernel has some remarkable properties. In particular, it gives information about the connection between a subset and its complement and allows us to characterize independent vertex sets.

We further study some relevant concepts both in the context of electrical network and in the context of reversible Markov Chains. Specifically, we study the effective resistance and the hitting time. Since we want to characterize these concepts for all the subsets of a network, and such concepts are obtained from the solution of suitable boundary value problems, we are focused on the analysis of the kernels that solve the corresponding boundary value problems. Hence in Section 3 we deal with the study of the Poisson kernel.

The authors gave in [3] a relation between the Poisson kernel and the normal derivative of the Green kernel, analogous to the continuous case. Here we obtain new and simple expressions for the Poisson and Green kernels in terms of equilibrium measures which, in particular, allow us to prove that the Laplacian verifies the condenser principle. In addition, we get some of the more relevant properties of these kernels. This link between kernels and equilibrium measures enables us to get explicit expressions for the effective resistance and for the hitting time in terms of equilibrium measures (Section 4). The expression of the Green function in terms of effective resistances (inverse resistive) has been proved by different authors by using distinct techniques, see for instance Coppersmith et al. [6], Metz [8] and Ponzio [9]. The results presented here allow us to obtain a straightforward proof of that relation.

Throughout the paper $\Gamma = (V, E, c)$ denotes an *electrical network*, that is, a simple and finite connected graph, with vertex set V and edge set E, in which each edge (x, y) has been assigned a *conductance* $c(x, y) > 0$. The *order of* Γ is $n = |V|$. Given a subset $F \subset V$, we denote by F^c its complement in V, and by $\delta(F) = \{x \in F^c : c(x, y) > 0 \text{ for some } y \in F\}$ and $\partial F = \{(x, y) \in E : x \in F, y \in F^c\}$ its *vertex boundary* and its *edge boundary*, respectively. We also use the notation $\overline{F} = F \cup \delta(F)$. The *Laplacian* of Γ is the matrix of order n whose entries are $\mathcal{L}(x, y) = -c(x, y)$ for all $x \neq y$ and $\mathcal{L}(x, x) = c(x)$, where $c(x) = \sum_{y\in V} c(x, y)$.

2 Potential Theory in finite spaces

Let V be a finite space with n points endowed with the discrete topology, and F be a non-empty subset of V. Then, the set of functions on V, denoted by $\mathcal{C}(V)$, and the set of non-negative functions on V, $\mathcal{C}^+(V)$, are naturally identified with $\mathbb{R}^n$ and the positive cone of $\mathbb{R}^n$, respectively. If $u \in \mathcal{C}(V)$, its *support* is the subset $\text{supp}(u) = \{x \in V : u(x) \neq 0\}$. Moreover, we consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \text{supp}(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$.

A symmetric function $\mathcal{K} : V \times V \longrightarrow \mathbb{R}$ will be called a *kernel* on V. Clearly, a kernel on V is identified with a real symmetric matrix of order n.

On the other hand, the set of Radon measures on V, denoted by $\mathcal{M}(V)$, is identified with $\mathcal{C}(V)$ and hence, if $\mu \in \mathcal{M}(V)$, its *support* is defined as above. Therefore, the sets of Radon measures supported by F, $\mathcal{M}(F)$, and of positive Radon measures supported by F, $\mathcal{M}^+(F)$, are identified with $\mathcal{C}(F)$ and $\mathcal{C}^+(F)$, respectively. In addition, if $\mu \in \mathcal{M}(V)$ its *mass* is given by $||\mu|| = \sum_{x\in V} \mu(x)$ and we denote by $\mathcal{M}^1(F)$, the set of positive Radon measures supported by F with unit mass. Finally, for each $x \in V$, ε_x stands for the Dirac measure on x, whereas the measure $\sum_{x\in F} \varepsilon_x$ will be denoted by $\mathbf{1}_F$. When $F = V$, the subscript in the above expression will be omitted.

If $\Gamma = (V, E, c)$ is a network then its Laplacian can be considered as a kernel on V. So, given $\mu \in \mathcal{M}(V)$ we will call *potential* and *energy of μ with respect to $\mathcal{L}$*, the function and the real number given respectively by

$$\mathcal{L}\mu(x) = \sum_{y\in V} c(x, y)\Big(\mu(x) - \mu(y)\Big) \quad x \in V \quad \text{and} \quad \mathcal{I}(\mu) = \langle \mathcal{L}\mu, \mu\rangle.$$

The application $\mathcal{I} : \mathcal{M}(V) \longrightarrow \mathbb{R}$ that assigns to each measure its energy, is a quadratic functional, whose associated bilinear form will be denoted by $\mathcal{I}(\cdot, \cdot)$ and is given by

$$\begin{aligned}\mathcal{I}(\mu_1, \mu_2) = \langle \mathcal{L}\mu_1, \mu_2\rangle &= \frac{1}{2}\sum_{x,y\in V} c(x, y)(\mu_1(x) - \mu_1(y))(\mu_2(x) - \mu_2(y)) \\ &= \langle \mu_1, \mathcal{L}\mu_2\rangle.\end{aligned}$$

Proposition 2.1. *The Laplacian kernel $\mathcal{L}$ satisfies the energy principle, i.e. $\mathcal{L}$ is strictly positive definite on* $\{\mu \in \mathcal{M}(V) : ||\mu|| = 0\}$.

Proof. Note that $\mathcal{I}(\mu) = \sum_{x,y\in V} c(x, y)(\mu(x) - \mu(y))^2 \geq 0$. Moreover, $\mathcal{I}(\mu) = 0$ iff $\mu = a\mathbf{1}$, $a \in \mathbb{R}$, since Γ is connected. □

Proposition 2.2. *The Laplacian kernel $\mathcal{L}$ satisfies Frostman's maximum principle, i.e.* $\max_{x\in V}\{\mathcal{L}\mu(x)\} = \max_{x\in \mathrm{supp}(\mu)}\{\mathcal{L}\mu(x)\}$, *for all $\mu \in \mathcal{M}^+(V)$.*

Proof. Let $\mu \in \mathcal{M}^+(V)$ and $F = \mathrm{supp}(\mu)$. If we consider $x \in F$ such that $\mu(x) = \max_{y\in F} \mu(y)$, then $\mathcal{L}\mu(x) \geq 0$. Moreover, for any $x \in F^c$, $\mathcal{L}\mu(x) \leq 0$. □

Proposition 2.3. *The Laplacian kernel satisfies the equilibrium principle, i.e., for every proper set $F \subset V$ there exists a unique $\nu^F \in \mathcal{M}^+(F)$ such that*

$$\mathcal{L}\nu^F = 1 \text{ on } F.$$

Moreover, $\mathrm{supp}(\nu^F) = F$.

Proof. It is known that if a kernel satisfies the energy and the maximum principles then

$$\min_{\mu\in\mathcal{M}^1(F)}\{\mathcal{I}(\mu)\} = \min_{\mu\in\mathcal{M}^1(F)} \max_{x\in V}\{\mathcal{L}\mu(x)\},$$

see, for instance, [7] for the general case and [3] for the discrete setting.

Moreover, the unique solution of the above problem, $\sigma \in \mathcal{M}^1(F)$, is such that $\mathcal{L}\sigma = \mathcal{I}(\sigma)$ in F and hence $\nu^F = \mathcal{I}(\sigma)^{-1}\,\sigma$.

To prove the last claim, suppose that there exists $x \in F$ such that $\nu^F(x) = 0$. Then $\mathcal{L}\nu^F(x) = -\sum_{y \in V} c(x, y)\nu^F(y) \le 0$, which contradicts $\mathcal{L}\nu^F = \mathbf{1}$ in F. □

For each proper set F the measure ν^F is called *the equilibrium measure for F.*

Next result is the well-known minimum principle for superharmonic functions. Here we give the proof for completeness and because monotonicity of equilibrium measures can be deduced straightforwardly from it.

Proposition 2.4. *Let F be a proper subset of V. The Laplacian $\mathcal{L}$, as an operator, verifies the minimum principle, i.e., if $u \in \mathcal{C}(\overline{F})$ is such that $\mathcal{L}(u) \ge 0$ on F, then*

$$\min_{x \in \delta(F)} \{u(x)\} \le \min_{x \in F} \{u(x)\}.$$

Proof. Let $m = \min_{x \in \delta(F)}\{u(x)\}$, and consider $w = u - m\mathbf{1}_{\overline{F}}$. Then, $\mathcal{L}w = \mathcal{L}u \ge 0$ on F and $w \ge 0$ on F^c. Let $x \in F$ be such that $w(x) = \min_{z \in F}\{w(z)\}$. To conclude, it is enough to show that $w(x) \le 0$ implies $w(x) = 0$.

Suppose that $w(x) \le 0$. Then, $w(x) \le w(z)$ for all $z \in V$ and therefore

$$0 \le \mathcal{L}w(x) = \sum_{z \in V} c(x, z)(w(x) - w(z)) \le 0,$$

which implies that $w(x) = w(z)$ for each $z \in V$ such that $c(x, z) \neq 0$. Of course, if $c(x, z) > 0$ for some $z \in F^c$, as $w(z) \ge 0$, necessarily $w(x) = 0$. Otherwise as Γ is connected and F is a proper set there exists $y \in F$ such that $w(y) = w(x)$ and $c(y, z) > 0$ for some $z \in F^c$, and hence $w(x) = 0$. □

Corollary 2.5. *If F and H are proper subsets such that $F \subset H$, then $\nu^F \le \nu^H$.*

The consideration of the Laplacian as a kernel in the context of Potential Theory allows us to introduce the Wiener capacity of a subset. We show that this concept is useful in order to obtain information about the connection between a subset of vertices and its complement. In [2] the authors developed an analogous approach in the case of graphs.

For each $F \subset V$, the value $I(F) = \inf_{\mu \in \mathcal{M}^1(F)} \mathcal{I}(\mu)$ is called the *energy of F*. Moreover, the value $\operatorname{cap}(F) = \frac{1}{I(F)}$ is known as the *Wiener capacity of F*. The unique measure $\sigma \in \mathcal{M}^1(F)$ such that $I(F) = \mathcal{I}(\sigma)$ is called *the capacitary measure for F*. It is easy to check that the Wiener capacity is a monotone set function, i.e., $\operatorname{cap}(F) \le \operatorname{cap}(H)$ when $F \subset H$. On the other hand, if F is a proper set and ν^F is its equilibrium measure, then $\mathcal{I}(\nu^F)^{-1} = ||\nu^F|| = \operatorname{cap}(F)$.

Proposition 2.6. *Let $F \subset V$ be such that $F = \bigcup_{i=1}^{s} F_i$, where F_i, $i = 1, \dots, s$, are the vertex sets of the connected components of the subnetwork induced by F. Then*

$$\operatorname{cap}(F) = \sum_{i=1}^{s} \operatorname{cap}(F_i).$$

Proof. If $F = V$, then $s = 1$, because Γ is connected. Hence, the result holds.

Suppose that F is a proper subset of V. For each $i = 1, \dots, s$, let μ^i be the capacitary measure for F_i. If $\beta = \sum_{i=1}^{s} \operatorname{cap}(F_i)$ and we consider $\mu \in \mathcal{M}^1(F)$ defined as

$$\mu = \frac{1}{\beta} \sum_{i=1}^{s} \operatorname{cap}(F_i)\mu^i,$$

then $\mathcal{L}\mu = \frac{1}{\beta} \sum_{i=1}^{s} \operatorname{cap}(F_i)\mathcal{L}\mu^i$. If $x \in F$, there exists k such that $x \in F_k$. Moreover, as $x \notin \overline{F_j}$ for all $j \neq k$, then $\mathcal{L}\mu^i(x) = I(F_k)$ if $i = k$ and $\mathcal{L}\mu^i(x) = 0$ otherwise. Hence, $\mathcal{L}\mu(x) = \frac{1}{\beta}$ for all $x \in F$, which implies that $I(F) = \frac{1}{\beta}$ and the result follows. □

The above result states that the Wiener capacity is additive with respect to the connected components of an induced subnetwork. However, it is not true for an arbitrary union of subsets of V. In fact, the following corollary shows that the Wiener capacity is not subadditive.

Corollary 2.7. *If $F \subset V$, then* $\operatorname{cap}(F) \geq \sum_{x \in F} \operatorname{cap}(\{x\})$. *Moreover, the equality holds iff F is an independent vertex set.*

Proof. Note that $I(\{x\}) = c(x)$, since $\mathcal{L}\varepsilon_x(x) = c(x)$. Consider the equilibrium measure ν^F for F, and let $\sigma(x) = \nu^F(x) - \frac{1}{c(x)}\mathbf{1}_F(x)$, $x \in V$. Then, for each $x \in F$

$$\mathcal{L}\sigma(x) = \sum_{y \in F} \frac{c(x, y)}{c(y)} \geq 0.$$

Hence, $\nu^F(x) \geq \frac{1}{c(x)}$ since $\mathcal{L}$ satisfies the minimum principle, which implies that $\operatorname{cap}(F) \geq \sum_{x \in F} \frac{1}{c(x)}$.

On the other hand, if F is an independent vertex set the equality follows from the above proposition. Conversely, suppose that F is a set of non independent vertices. Then there exist $x_0, y_0 \in F$ such that $c(x_0, y_0) > 0$ which implies that $\mathcal{L}\sigma(x_0) > 0$ and hence $\nu^F(x_0) > \frac{1}{c(x_0)}$. □

The Wiener capacity for the Laplacian kernel is not subadditive due to the fact that the Laplacian is not a positive kernel. If $q = \max_{x,y \in V}\{c(x, y)\}$ and $F \subset V$, the value $(I(F) + q)^{-1}$ can be seen as the Wiener capacity of F with respect to the positive kernel $\mathcal{L} + qJ$, where J denotes the kernel whose values are equal to one. In fact, the Wiener capacity is subadditive for a positive kernel, see [7, p. 157].

Proposition 2.8. *Let* $F_1, \dots, F_s \subset V$ *and* $F = \bigcup_{i=1}^{s} F_i$. *Then*

$$(I(F)+q)^{-1} \leq \sum_{i=1}^{s} (I(F_i)+q)^{-1}.$$

In particular, we have the following result.

Corollary 2.9. *Let* $F \subset V$ *be a proper subset. Then* $\operatorname{cap}(F)\operatorname{cap}(F^c) \geq \frac{1}{q^2}$.

Before ending this section let us determine the Wiener capacities and the capacitary measures for connected proper subsets of a weighted path which will help us to study the sharpness of the lower bound in the above corollary.

Given a path P_n with $n \geq 2$ vertices and conductances c_{ii+1}, $i = 1, \dots, n-1$, consider a proper subset $F = \{x_1, \dots, x_s\}$ of P_n. Then $\sigma(x_j) = \sum_{i=j}^{s} \frac{i}{c_{ii+1}}$ for any $x_j \in F$, and $\operatorname{cap}(F) = \sum_{i=1}^{s} \frac{i^2}{c_{ii+1}}$. By considering $p = \min_{(x,y)\in E}\{c(x,y)\}$, we get that

$$\frac{1}{6q^2}(n-1)n(2n-1) \leq \operatorname{cap}(F)\operatorname{cap}(F^c) \leq \frac{1}{24^2 p^2} n^2(n+1)^2(n+2)^2.$$

On the other hand, in a complete network in which each edge has conductance q, the product of capacities attains its minimum value for any subset. The differences in the behaviour of the capacity products are due to the different degrees of connection between the vertices of F and F^c. In particular, the following result characterizes when equality holds in Corollary 2.9.

Proposition 2.10. *Let* $F \subset V$ *be a proper subset. Then,*

$$\operatorname{cap}(F)\operatorname{cap}(F^c) = \frac{1}{q^2} \iff |\partial F| = |F||F^c| \text{ and } c(x,y) = q,\ \forall (x,y) \in \partial F.$$

Moreover, the capacitary measures for F *and* F^c *are the uniform measures on* F *and* F^c *respectively.*

Proof. Note that $|\partial F| = |F||F^c|$ and $c(x,y) = q$, $\forall (x,y) \in \partial F$ iff $\sum_{y\in F^c} c(x,y) = q|F^c|$ for all $x \in F$ and $\sum_{x\in F} c(x,y) = q|F|$ for all $y \in F^c$. In addition, the uniform measures on F and F^c, $\mu_1 = \frac{1}{|F|}\mathbf{1}_F$ and $\mu_2 = \frac{1}{|F^c|}\mathbf{1}_{F^c}$, satisfy $\mathcal{L}\mu_1 = \frac{q|F^c|}{|F|}$ on F and $\mathcal{L}\mu_2 = \frac{q|F|}{|F^c|}$ on F^c, respectively. Therefore, they are the capacitary measures for F and F^c, and $\operatorname{cap}(F)\operatorname{cap}(F^c) = \frac{1}{q^2}$.

Conversely, if $\mathcal{K} = \mathcal{L} + qJ$ and we consider $\mathbf{1} = \mathbf{1}_F + \mathbf{1}_{F^c}$, then

$$\begin{aligned} qn^2 &= \langle \mathcal{K}\mathbf{1}, \mathbf{1}\rangle = \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_F\rangle + \langle \mathcal{K}\mathbf{1}_{F^c}, \mathbf{1}_{F^c}\rangle + 2\langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c}\rangle \\ &\geq (\mathcal{I}(\mathbf{1}_F) + q|F|^2) + (\mathcal{I}(\mathbf{1}_{F^c}) + q|F^c|^2) \\ &\geq |F|^2(I(F)+q) + |F^c|^2(\mathcal{I}(F^c)+q) \\ &\geq \frac{(|F|+|F^c|)^2}{\frac{1}{I(F)+q} + \frac{1}{I(F^c)+q}} = \frac{n^2}{\frac{1}{I(F)+q} + \frac{1}{I(F^c)+q}} \, . \end{aligned}$$

On the other hand, $\operatorname{cap}(F)\operatorname{cap}(F^c) = \frac{1}{q^2}$ iff $\frac{1}{(I(F)+q)} + \frac{1}{(I(F^c)+q)} = \frac{1}{q}$. Therefore, by using the above inequalities, we conclude that

$$\operatorname{cap}(F)\operatorname{cap}(F^c) = \frac{1}{q^2} \implies \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c}\rangle = 0.$$

Finally, it is enough to observe that

$$|\partial F| = |F||F^c| \text{ and } c(x,y) = q,\ \forall (x,y) \in \partial F \text{ iff } \langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c}\rangle = 0,$$

because $\langle \mathcal{K}\mathbf{1}_F, \mathbf{1}_{F^c}\rangle = q|F||F^c| - \sum_{(x,y)\in\partial F} c(x,y)$. □

3 The Poisson Kernel

In this section we consider a semihomogeneous Dirichlet Problem, and we solve it by considering the associated Poisson Kernel. Our purpose is to use such a kernel for finding an expression of the effective resistance between subsets, in the next section.

Let $F \subset V$ be a proper subset. A *Semihomogeneous Dirichlet Problem* on F consists in finding, given $g \in \mathcal{C}(\delta(F))$, a function $u \in \mathcal{C}(\overline{F})$ such that

$$\left.\begin{aligned} \mathcal{L}u(x) &= 0, && x \in F,\\ u(x) &= g(x), && x \in \delta(F). \end{aligned}\right\} \tag{3.1}$$

As shown in [3], this problem has a unique solution which can be obtained by means of the Poisson Kernel.

A function $P : \overline{F} \times \delta(F) \longrightarrow \mathbb{R}$ is called the *Poisson kernel* for Problem (3.1) if for all $y \in \delta(F)$, $P_y = P(\cdot, y)$ is the solution of the following problem:

$$\left.\begin{aligned} \mathcal{L}P_y &= 0, && \text{in } F,\\ P_y &= \varepsilon_y, && \text{in } \delta(F). \end{aligned}\right\} \tag{3.2}$$

From the definition, it is easy to check that the function $u(x) = \sum_{y\in\delta(F)} P(x,y)g(y)$ is the solution of Problem (3.1). Now we show how the use of equilibrium measures allow us to obtain an expression of the Poisson Kernel for every proper set F such that $|\delta(F)| \geq 2$.

Proposition 3.1. *Let $F \subset V$ be a proper set. If $F = V \setminus \{y\}$, then $\delta(F) = \{y\}$ and $P_y = \mathbf{1}$. Otherwise, the Poisson Kernel for F is given by:*

$$P(x,y) = \left(\nu^{F\cup\{y\}}(y)\right)^{-1}\left(\nu^{F\cup\{y\}}(x) - \nu^F(x)\right).$$

As the Poisson kernel for Problem (3.1) only depends on F, we will call it the Poisson kernel for F and it will be denoted by P^F.

Corollary 3.2 (The condenser principle). *Let F be a proper subset of V and $\{A, B\}$ a partition of $\delta(F)$. Then, $u \in \mathcal{C}(\overline{F})$, the unique solution of the boundary value problem*

$$\left.\begin{aligned} \mathcal{L}u(x) &= 0 \quad \text{if } x \in F,\\ u(x) &= 1 \quad \text{if } x \in A,\\ u(x) &= 0 \quad \text{if } x \in B, \end{aligned}\right\}$$

is such that $0 \le u \le 1$ on V, $\mathcal{L}u \ge 0$ on A, and $\mathcal{L}u \le 0$ on B.

Proof. From Proposition 3.1, the solution of the boundary value problem is

$$u = \sum_{y \in A} \frac{\nu^{F\cup\{y\}} - \nu^F}{\nu^{F\cup\{y\}}(y)}.$$

Moreover, from Corollary 2.5, $u \ge 0$ on V, which implies that if $x \in B$ then $\mathcal{L}u(x) = -\sum_{z\in V} c(x,z)u(z) \le 0$. Consider now $v = \mathbf{1} - u$, then v is the solution of

$$\left.\begin{aligned} \mathcal{L}v(x) &= 0 \quad \text{if } x \in F,\\ v(x) &= 0 \quad \text{if } x \in A,\\ v(x) &= 1 \quad \text{if } x \in B. \end{aligned}\right\}$$

Therefore, reasoning as above, $v \ge 0$ on V, $\mathcal{L}v \le 0$ on A and *a fortiori* $u \le \mathbf{1}$ and $\mathcal{L}u \ge 0$ on A. □

Proposition 3.3. *If F is a proper subset of V, then $0 \le P_y^F \le 1$ for each $y \in \delta(F)$. Moreover, if $F \ne V - \{y\}$ then $\mathcal{L}P_y^F(y) > 0$, and when F is connected, $0 < P_y^F < 1$ on F for each $y \in \delta(F)$.*

Proof. If $y \in \delta(F)$ and we consider $u = P_y^F$, then u satisfies $\mathcal{L}u = 0$ on F, $u(y) = 1$ and $u = 0$ on $\delta(F) - \{y\}$. Then, applying the condenser principle with $A = \{y\}$ and $B = \delta(F) - \{y\}$, we get that $0 \le u \le 1$ and moreover $\mathcal{L}u(y) \ge 0$. As $\langle \mathcal{L}(u), u\rangle = \mathcal{L}u(y)$, it follows that $\mathcal{L}u(y) = 0$ iff $u = a\mathbf{1}$, $a \in \mathbb{R}$. Moreover, if $F \ne V \setminus \{y\}$, $a = 0$ since $u = 0$ on $\delta(F) \setminus \{y\}$. So, in this case $\mathcal{L}u(y) > 0$. On the other hand, if $x \in F$, necessarily $u(x) > 0$, since otherwise

$$0 = \mathcal{L}(\nu^{F\cup\{y\}} - \nu^F)(x) = -\sum_{z\in V} c(x,z)\Big(\nu^{F\cup\{y\}}(z) - \nu^F(z)\Big) \le 0,$$

because of Corollary 2.5, and therefore $\nu^{F\cup\{y\}}(z) = \nu^F(z)$ for all z such that $c(x,z) \ne 0$. Since Γ is connected, we get that $\nu^{F\cup\{y\}}(y) = \nu^F(y) = 0$, which is a contradiction. Reasoning analogously for $v = 1 - u$, we get that $u < 1$ for all $z \in F$. □

It is well known that a Dirichlet Problem can be solved by using the associated Green kernel, so that there should be a relation between the Poisson and Green kernels.

This relation is well known in the continuous case and we investigate it next for the discrete case. Let us start by defining the Green kernel for a subset.

A function $G : \overline{F} \times F \longrightarrow \mathbb{R}$ is called the *Green kernel* for Problem (3.1) if for all $y \in F$, $G_y = G(\cdot, y)$ is the solution of the following problem:

$$\left.\begin{aligned} \mathcal{L}G_y &= \varepsilon_y \quad \text{in } F,\\ G_y &= 0 \quad \text{in } \delta(F). \end{aligned}\right\} \tag{3.3}$$

Clearly the solution of (3.1) is given by $u(x) = g(x) - \sum_{y\in F} G(x,y)\mathcal{L}g(y)$. As before, the Green kernel depends only of F so that it will be denoted by G^F.

Proposition 3.4. *Let F be a proper subset of V. Then*

(i) *G^F is a symmetric kernel, and $G^F_y(y) > 0$, $P^{F-\{y\}}_y = \frac{G^F_y}{G^F_y(y)}$ and $0 \le G^F_y \le G^F_y(y)$ for each $y \in F$.*

can you reformulate to avoid this overfull line?

(ii) *$P^F(x,y) = \varepsilon_x(y) - \frac{\partial}{\partial \eta_y} G^F(x,y)$, where $\frac{\partial}{\partial \eta_y} G^F(x,y) = -\sum_{z\in V} c(y,z) G^F(x,z)$ denotes the normal derivative of $G^F(x,\cdot)$.*

(iii) *$G^F(x,y) = \frac{\nu^F(y)}{||\nu^F|| - ||\nu^F_y||}\Big(\nu^F(x) - \nu^F_y(x)\Big)$, where ν^F_y denotes the equilibrium measure for the set $F - \{y\}$, $y \in F$.*

Proof. (i) Clearly, G^F is symmetric and non-negative from Proposition 2.4. On the other hand, if $y \in F$, then $G^F_y(y) > 0$ since otherwise we would have $\mathcal{L}G^F_y(y) = -\sum_{z\in V} c(y,z) G^F_y(z) \le 0$, in contradiction with $\mathcal{L}G^F_y = \varepsilon_y$. Moreover, if we consider $u = \frac{1}{G^F_y(y)} G^F_y$, it follows that $u(y) = 1$, $u = 0$ on $\delta(F)$ and in addition $\mathcal{L}u = 0$ on $F\setminus\{y\}$. Therefore, $u = P^{F\setminus\{y\}}_y$. The last part follows from Proposition 3.3.

(ii) For each $y \in \delta(F)$ and each $x \in \overline{F}$, let $u(x) = \varepsilon_x(y) - \frac{\partial}{\partial \eta_y} G^F(x,y)$. Then $\mathcal{L}u = 0$ on F, $u(y) = 1$ and $u = 0$ on $\delta(F) \setminus \{y\}$. Hence $u = P^F_y$.

(iii) From (i), $G^F_y = G^F_y(y) P^{F\setminus\{y\}}_y$. It is enough to find the value $G^F_y(y)$ by imposing the conditions satisfied by the Green kernel. So,

$$\begin{aligned} 1 = \mathcal{L}G^F_y(y) &= \frac{G^F_y(y)}{\nu^F(y)}\big(\mathcal{L}\nu^F(y) - \mathcal{L}\nu^F_y(y)\big)\\ &= \frac{G^F_y(y)}{\nu^F(y)}\big(1 - \mathcal{L}\nu^F_y(y)\big) = \frac{G^F_y(y)}{\nu^F(y)^2}\big(||\nu^F|| - ||\nu^F_y||\big), \end{aligned}$$

since $||\nu^F_y|| = \langle \nu^F_y, \mathbf{1}\rangle = \langle \nu^F_y, \mathcal{L}\nu^F\rangle = \langle \mathcal{L}\nu^F_y, \nu^F\rangle = ||\nu^F|| - \nu^F(y)\big(1 - \mathcal{L}\nu^F_y(y)\big)$. $\square$

4 The effective resistance

The effective resistance between two vertices or two subsets of a network is defined as the inverse of the current arising from applying to them the unit potential. It is well known that the effective resistance is the inverse of the energy of the solution of the Dirichlet problem. It is also known that we can restrict ourselves to the case in which the source and the sink are single vertices joined with an edge. Here we look (similarly to the continuous case [1]) at the situation when the vertex boundary of a set F is partitioned into three subsets: two of them are the source (one vertex) and the sink (one vertex), and the remaining part of the boundary is insulated. The general effective resistance can be defined in terms of the energy of the solution of the corresponding mixed boundary value problem which is solved by using the Poisson kernel. Then, using equilibrium measures, we deduce an expression for the effective resistance between two vertices of an electrical network when the remaining part of the boundary is insulated. If the insulated subset is empty, then we obtain a formula for the standard effective resistance. We also examine the probabilistic interpretation of these results.

Let $F \subset V$ and $\delta(F) = \{y\} \cup \{z\} \cup D$, a partition of the vertex boundary. If the unit potential is applied across vertices y and z, then, according to Kirchhoff's Laws, the potential u at vertices of the network has to be the solution of the following *mixed boundary value problem*:

$$\left.\begin{aligned} \mathcal{L}u(x) &= 0 \quad \text{if} \quad x \in F, \\ u(y) &= 1, \\ u(z) &= 0, \\ \frac{\partial u}{\partial \eta}(x) &= 0 \quad \text{if} \quad x \in D, \end{aligned}\right\} \tag{4.4}$$

where $\frac{\partial u}{\partial \eta}(x) = \sum_{t \in F} c(x,t)\big(u(x) - u(t)\big)$ is the *normal derivative* of u at a point $x \in D$. When $D = \emptyset$, Problem (4.4) gives the standard notion of effective resistance between vertices y and z. From now on we suppose that $F \cup \{y\} \cup \{z\} \cup D = V$, because the potentials at vertices of the set $V \setminus (F \cup \{y\} \cup \{z\} \cup D)$ are zero and hence these vertices can be identified with z.

On the other hand, as shown in [3], this problem has a unique solution and can be transformed into a Dirichlet problem in the following way: consider a new network built from the subnetwork induced by F adding its edge and vertex boundaries. Specifically, given F, we define the network $\overline{\Gamma}(F) = (\overline{F}, \overline{E}, \overline{c})$, where $\overline{E} = \{(x,t) \in E : x \in F\}$, and the conductance function $\overline{c}$ is the restriction of c to $\overline{E}$. We denote the Laplacian of this network by $\overline{\mathcal{L}} = \mathcal{L}(\overline{\Gamma})$. Note that $\overline{\mathcal{L}}u(x) = \mathcal{L}u(x)$ if $x \in F$, and $\overline{\mathcal{L}}u(x) = \frac{\partial u}{\partial \eta}(x)$ if $x \in \delta(F)$. So, $u \in \mathcal{C}(\overline{F})$ is the solution of Problem (4.4) iff u is the solution of the following Dirichlet problem:

$$\left.\begin{aligned} \overline{\mathcal{L}}u(x) &= 0,\ x \in F \cup D, \\ u(y) &= 1, \\ u(z) &= 0. \end{aligned}\right\} \qquad (4.5)$$

We define the *effective resistance between y and z when D is insulated* as $R^D_{yz} = \overline{\mathcal{I}}(u)^{-1}$, where $\overline{\mathcal{I}}$ is the energy with respect to the kernel $\overline{\mathcal{L}}$. As $u(x) = P^{F\cup D}(x, y)$, to find an expression for the effective resistance we must know the value of $P^{F\cup D}_y$. Let us see that in this case we can get a simpler expression for that kernel. To simplify the notation, we will write P^{yz} instead of $P^{V\setminus\{y,z\}}$.

Proposition 4.1. *Let $F \cup D = V \setminus \{y, z\}$. Then the Poisson kernel for $F \cup D$ is given by:*

$$P^{yz}(x, y) = \frac{\nu_z(x) - \nu_y(x) + \nu_y(z)}{\nu_z(y) + \nu_y(z)} \quad \textit{and} \quad P^{yz}(x, z) = \frac{\nu_y(x) - \nu_z(x) + \nu_z(y)}{\nu_z(y) + \nu_y(z)},$$

where ν_t denotes the equilibrium measure for the set $V\setminus\{t\}$ with respect to the kernel $\overline{\mathcal{L}}$.

Proof. Consider the unique solutions $u, v \in \mathcal{C}(V)$ of the problems

$$\left.\begin{aligned} \overline{\mathcal{L}}u(x) &= 0,\ x \in F \cup D \\ u(y) &= 1 \\ u(z) &= 0 \end{aligned}\right\} \quad \text{and} \quad \left.\begin{aligned} \overline{\mathcal{L}}v(x) &= 0,\ x \in F \cup D \\ v(y) &= 0 \\ v(z) &= 1 \end{aligned}\right\}$$

respectively. Then, both solutions are determined by the identities

$$u(x) = P^{yz}(x, y) \quad \text{and} \quad v(x) = P^{yz}(x, z).$$

Keeping in mind the expression for P^{yz} given in Proposition 3.1, we obtain that $u = \frac{\nu_z - \nu_{yz}}{\nu_z(y)}$ and $v = \frac{\nu_y - \nu_{yz}}{\nu_y(z)}$, respectively, where ν_{yz} denotes the equilibrium measure for the set $V \setminus \{y, z\}$. On the other hand, as $u + v = \mathbf{1}$, adding the above expressions gives

$$\nu_{yz} = \frac{\nu_z \nu_y(z) + \nu_y \nu_z(y) - \nu_y(z)\nu_z(y)}{\nu_y(z) + \nu_z(y)}.$$

The result then follows from substituting the above expression in the formulas for $P^{yz}(x, y)$ and $P^{yz}(x, z)$. □

Corollary 4.2. *Let F be a subset of V such that $\delta(F) = \{y\} \cup \{z\} \cup D$. Then*

$$R^D_{yz} = \frac{\nu_y(z) + \nu_z(y)}{n}.$$

Proof. Let u be the solution of problem (4.5). Then, keeping in mind the expression for the Poisson kernel of the set $V \setminus \{y, z\}$ obtained in Proposition 4.1, we get that

$$\overline{\mathfrak{I}}(u) = \overline{\mathcal{L}}u(y) = \frac{1}{\nu_y(z) + \nu_z(y)}\overline{\mathcal{L}}(\nu_z - \nu_y)(y) = \frac{n}{\nu_y(z) + \nu_z(y)},$$

because $0 = \langle \nu_y, \overline{\mathcal{L}}\mathbf{1}\rangle = \langle \overline{\mathcal{L}}\nu_y, \mathbf{1}\rangle = n - 1 + \overline{\mathcal{L}}\nu_y(y)$. □

Corollary 4.3. *Let $z \in V$. If G^z denotes the Green kernel for the set $V \setminus \{z\}$, then $G^z(y, y) = R_{yz}$. Moreover,*

$$G^z(x, y) = \frac{1}{n}\big(\nu_z(x) - \nu_y(x) + \nu_y(z)\big).$$

Proof. From the proof of Proposition 3.4 (iii), we get that

$$G^z(y, y) = \frac{\nu_z(y)}{1 - \mathcal{L}\nu_{yz}(y)}.$$

On the other hand, keeping in mind the expression for ν_{yz} obtained in the proof of Proposition 4.1, we get $\mathcal{L}\nu_{yz}(y) = \frac{1}{\nu_y(z)+\nu_z(y)}(\nu_y(z) + \nu_z(y)(1 - n)) = 1 - \frac{\nu_z(y)}{R_{yz}}$. Hence, $G^z(y, y) = R_{yz}$. Moreover from Proposition 3.4 (i),

$$G^z(x, y) = \frac{\nu_z(x) - \nu_y(x) + \nu_y(z)}{\nu_z(y) + \nu_y(z)}R_{yz} = \frac{1}{n}\big(\nu_z(x) - \nu_y(x) + \nu_y(z)\big). \qquad □$$

We can obtain direct formulas for the Poisson kernel of the set $V \setminus \{y, z\}$ and for the Green kernel for the set $V \setminus \{z\}$ in terms of the effective resistance between two vertices.

Proposition 4.4. *Let $y, z \in V$, then*

(i) *The Green kernel for the set $V \setminus \{z\}$ is given by*

$$G^z(x, y) = \frac{1}{2}\big(R_{xz} + R_{yz} - R_{xy}\big).$$

(i) *The Poisson kernel for the set $V \setminus \{y, z\}$ is given by*

$$P^{yz}(x, y) = \frac{1}{2R_{yz}}\big(R_{xz} + R_{yz} - R_{xy}\big), \quad P^{yz}(x, z) = \frac{1}{2R_{yz}}\big(R_{xy} + R_{yz} - R_{xz}\big).$$

Proof. (i) From Corollary 4.3 we know that

$$\begin{aligned} G^z(x, y) &= \tfrac{1}{n}\big(\nu_z(x) - \nu_y(x) + \nu_y(z)\big) \\ &= \tfrac{1}{2n}\big(\nu_z(x) - \nu_y(x) + \nu_y(z) + \nu_z(y) - \nu_x(y) + \nu_x(z)\big) \\ &= \tfrac{1}{2}\big(R_{xz} + R_{yz} - R_{xy}\big), \end{aligned}$$

where the second identity follows from the symmetry of the Green kernel.

(ii) It follows from the previous point and from part (i) of Proposition 3.4. □

The expression obtained in the above proposition for the Green kernel is well known in the context of the so-called *resistive inverse*. Specifically, given the matrix $(R_{xy})_{x,y\in V}$ of effective resistances of a network, we are interested in finding the matrix of conductances of the network. Coppersmith et al. [6] gave a simple but obscure four-step algorithm for computing the resistive inverse. Later, Ponzio [9] gave a self-contained combinatorial explanation of this algorithm. This relation was also given by Metz [8] using Dirichlet forms. Here we have obtained a new and simple proof of that algorithm in term of equilibrium measures.

Some of the concepts considered here have a well-known probabilistic interpretation. For instance, the effective resistance is related with the escape probability for a reversible Markov chain. Also, Problem 4.4 can be described in terms of the Neumann random walk, see [5]. Hence, the general concept of effective resistance corresponds to a generalization of the escape probability.

Finally, let us consider another Dirichlet problem whose solution has an important probabilistic meaning. For that, let $\Gamma = (V, E, c)$ be the network that has as vertices the states of a reversible Markov chain and as conductances $c(x, y) = \pi(x)p(x, y)$, where $p(x, y)$ is the transition probability from state x to state y and $\pi(x)$ is the value of the stationary distribution at state x. Then, the *hitting time $H(x, y)$ from x to y*, defined as the expected number of steps in order to reach the state y from the state x, satisfies the following relations:

$$\begin{aligned} \mathcal{L}H_y(x) &= c(x), \quad x \in V \setminus \{y\},\\ H_y(y) &= 0. \end{aligned}$$

Therefore, by using the expression for the Green kernel G^y we obtain

$$H(x, y) = \frac{1}{n}\sum_{z\in V} c(z)\big(\nu_z(y) + \nu_y(x) - \nu_z(x)\big),$$

and also the well known relation between the hitting time and the effective resistance, see [10]:

$$H(x, y) = \frac{1}{2}\sum_{z\in V} c(z)\big(R_{xy} + R_{yz} - R_{xz}\big).$$

Acknowledgement. This work has been supported by the Comisión Interministerial de Ciencia y Tecnología (Spanish Research Council) under project BFM2000-1063 and the ETSECCPB.

References

[1] L. V. Ahlfors and L. Sario, Riemann Surfaces, Princeton Univ. Press, Princeton, NJ, 1960.

[2] E. Bendito, A. Carmona and A. M. Encinas, Shortest paths in distance-regular graphs, Europ. J. Combin. 21 (2000), 153–166.

[3] E. Bendito, A. Carmona and A. M. Encinas, Solving boundary value problems on networks using equilibrium measures, J. Funct. Anal. 171 (2000), 155–176.

[4] G. Choquet, J. Deny, Modèles finis en théorie du potentiel, J. Analyse Math. 5 (1956/57), 77–135.

[5] F. R. K. Chung, Spectral Graph Theory, CBMS Regional Conference Series in Mathematics, 92, Amer. Math. Soc., Providence, RI, 1997.

[6] D. Coppersmith, P. Doyle, P. Raghavan and M. Snir, Random walks on weighted graphs and applications to on-line algorithms, J. Assoc. Comput. Mach. 40 (1993), 421–453.

[7] B. Fuglede, On the theory of potentials in locally compact spaces, Acta Math. 103 (1960), 139–215.

[8] V. Metz, Shorted operators: an application in potential Theory, Linear Algebra Appl. 264 (1997), 439–455.

[9] S. Ponzio, The combinatorics of effective resistances and resistive inverses, Inform. and Comput. 147 (1998), 209–223.

[10] P. Tetali, Random walks and the effective resistance of networks, J. Theoret. Probab. 4 (1991), 101–109.

Enrique Bendito, Angeles Carmona, Andrés M. Encinas, Departament de Matemàtica Aplicada III, Universitat Politècnica de Catalunya, 08034 Barcelona, Spain

E-mail: enrique.bendito@upc.es
angeles.carmona@upc.es
andres.marcos.encinas@upc.es